Solving integral algebraic equations of index 1

Jalil Manafian
Department of Mathematics, College of Mathematics, Ahar Branch, Islamic Azad university, Ahar, Iran
Isa Zamanpour
Department of Mathematics, College of Mathematics, Karaj Branch, Islamic Azad university, Karaj, Iran

Abstract:
In this paper, we use continuous collocation methods for IAEs and we give a comprehensive analysis for (order of) convergence and divergence of continuous collocation methods in solving integral algebraic equations of index 1. We compare the extended index notations from DAEs to IAEs and we show that all introduced index for index 1 case are same. To simplify the analysis, we state some lemmas to connect the integral algebraic equations to the first kind Volterra integral equations. The results of numerical experiments support the theoretical results.

Keywords: Continuous collocation methods, Volterra integral equations, Integral algebraic equations, Integral algebraic index

Corresponding Author: Jalil Manafian
Email: j-manafian@iau-ahar.ac.ir
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1. Introduction

In this paper we consider Integral Algebraic Equations (IAEs) of the form

\[ A(x)y(x) + \int_0^x k(x,s)y(s)\,ds = g(x), \quad t \in I := [0,T]. \]  

(1)

and the nonlinear form

\[ A(x)y(x) + \int_0^x k(x,s,y(s))\,ds = g(x), \quad t \in I := [0,T], \]  

(2)

where \( A \in C(I, R^{r \times r}) \), \( g \in C(I, R^r) \), \( k \in C(D, R^{r \times r}) \) and \( K \in C(D \times R^r, R^r) \), with \( D := \{(x,s): 0 \leq s \leq x \leq T\} \). We analyze the linear form and extend same results to the nonlinear form. If \( A \) is nonsingular for all \( t \in I \) then multiplying (1) by \( A^{-1} \) changes it to a system of second kind Volterra integral equations that its theoretical and numerical analysis have been almost completely investigated [1]. Hence, we assume that \( A(t) \) is a singular matrix with constant rank for all \( t \in I \).

There are different notions of index for classification of IAEs. For example, Gear introduced differentiation index for IAEs [2]. Left index for (1) is another notion that was used firstly by Russian mathematicians [3,4]. We similarly define operator index, which makes our analysis comprehensive and easy.

In this paper, we investigate the continuous piecewise polynomial collocation method for IAEs of operator index 1, in new sense, and analyze convergence, divergence and order of convergence. In spite of the fact that IAEs are ‘mixed’ system of the first and second kind Volterra integral equations, this analysis is not as easy as for the first and second kind Volterra integral equations. For example, in the convergence analysis of the first kind Volterra integral equations with kernel \( k \), it is necessary some equations to be divided by scalar \( k(x,x) \neq 0 \) in the mesh points, which in the system case, it means that the matrix function \( k \) must be invertible on its domain, while in an IAE it does not generally hold. Hence we are interested in connecting the results of [5, 6] to IAEs which simplify our analysis.

\[ y(x) + K_{11}y(x) + K_{12}z(x) = q_1(x), \]  

(3)

\[ K_{21}y(x) + K_{22}z(x) = q_2(x), \]  

(4)

The piecewise polynomial collocation methods for solving IAEs of differentiation index 1 have been investigated by Kauthen [7]. He showed that the order of error for these methods is \( m \) and \( m-1 \) if the stability function \( R(\infty) = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{s_i} \) satisfies respectively in \( R(\infty) \in [-1, 1) \) and \( R(\infty) = 1 \), where \( c_i, i = 1, \cdots, m \) are the collocation parameters. The analysis of the continuous piecewise polynomial collocation methods are more complicated than those for the piecewise polynomial collocation methods. We show that there exist collocation parameters for which the continuous piecewise polynomial collocation method is divergent. This paper is organized as follows: in section 2, we give a brief review of preliminary results and statements for simplifying analysis and proving uniqueness and existence theorems for the solution of (1). In section 3, we apply the continuous piecewise polynomial collocation method for the system (1). An existence and uniqueness theorem for the approximate solution is proved in section 4. In section 5, we introduce generalized difference inequalities for supporting our analysis. In section 6, a global convergence theorem for \( c_m = 1 \) is proved. In section 7, a divergence theorem is stated. In section 8, a global convergence theorem for \( c_m < 1 \) is proved. Finally, in section 9, we illustrate the results of paper by giving some numerical experiments.
2. Preliminary

In this section we recall preliminary results and construct some important techniques.

Definition 1- The matrix $A^{-}(x)$ is called semi-inverse matrix for $A(t)$ if it satisfies the equation

$$A(x)A^{-}(x)A(x) = A(x),$$

which can be rewritten as

$$V(x)A(x) = 0,$$

with

$$V(x) = I - A(x)A^{-}(x),$$

where $I$ is and $r 	imes r$ identity matrix. The following conditions are necessary and sufficient for the existence of a semi-inverse matrix $A^{-}(x)$ with elements in $C([0,1], R^{r 	imes r})$:

1. the elements of $A(x)$ belong to $C([0,1], R^{r 	imes r})$,
2. rank $(x) = \text{const}$, $\forall x \in [0,1]$.

Definition 2- The matrix pencil $\lambda A(x) + k(x,x)$ satisfies the “rank-degree” criterion on the interval $[0,1]$, if rank $A(x) = \text{deg det}(\lambda A(x) + k(x,x)) = \text{const}$, for all $x \in [0,1]$.

Lemma 1- Let the matrix pencil $\lambda A(x) + k(x,x)$ satisfy the “rank-degree” criterion on the interval $[0,1]$. Then $\text{det}(A(x) + V(x)k(x,x)) \neq 0$, and $\text{det}(A(x) + V(x)(A(x) + k(x,x))) \neq 0$, $\forall x \in [0,1]$.

Lemma 2- Let the matrix pencil $\lambda A(x) + k(x,x)$ satisfy the “rank-degree” criterion on the interval $[0,1]$. Then there exists an invertible matrix $P(x)$ such that

$$A_p = PAA^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \quad A_p(x) + V_p(x)k_p(x,x) = \begin{pmatrix} A_1 & A_2 \\ K_1 & K_2 \end{pmatrix},$$

where $A_1$ and $-K_1A^{-1}A_2 + K_2$ are invertible.

By introducing the notations

$$L = (l_{ij}), \quad L = (l_{ij}), \quad L^{-1}L = (l_{ij}), \quad N_c = \begin{pmatrix} cA_1 \\ K_1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $c$ is a non-zero constant, we have

$$\theta N_c^{-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \theta N_c^{-1}v = \theta v.$$ 

there exists an invertible matrix $P_1$ independent of $c$ such that

$$P_1N_c^{-1}vP_1^{-1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. $$

In what follows, especially in the proof of convergence and divergence theorems, we will deal with a matrix of the form

$$R = G^{-1}H,$$
where

\[
G = \begin{bmatrix}
L_{1,1}N_1 & L_{1,2}m & \cdots & L_{1,m}m \\
\vdots & \vdots & & \vdots \\
L_{m,1}m & L_{m,2}m & \cdots & L_{m,m}N_1
\end{bmatrix}, \quad H = \begin{bmatrix}
L_{1,1}N_1 & L_{1,2}m & \cdots & L_{1,m}m \\
\vdots & \vdots & & \vdots \\
L_{m,1}m & L_{m,2}m & \cdots & L_{m,m}N_1
\end{bmatrix}, \quad (8)
\]

Applying the first stage of Gaussian elimination on the matrix \([G,H]\) and using the relation

\[
L_{j,j}N_1 - \frac{L_{j,1}L_{1,j}}{L_{1,1}} mN_1^{-1} m = L_{j,j}N_1 - \frac{L_{j,1}L_{1,j}}{L_{1,1}} m = \frac{L_{1,1}L_{j,j} - L_{j,1}L_{1,j}}{L_{1,1}} N_{c_2},
\]

We have

\[
R = \begin{bmatrix}
I & \frac{L_{1,2}N_1^{-1} \theta \cdots \frac{L_{1,m}N_1^{-1} \theta m}{L_{1,1}}}{L_{1,1}} \\
0 & \frac{L_{1,1}L_{j,j} - L_{1,j}L_{j,1}}{L_{1,1}} N_{c_2} \cdots \frac{L_{1,1}L_{2,m} - L_{1,m}L_{2,1}}{L_{1,1}} \theta \cdots \frac{L_{1,1}L_{m,1} - L_{1,1}L_{1,m}}{L_{1,1}} \theta \\
0 & \frac{L_{1,1}L_{m,2} - L_{1,2}L_{m,1}}{L_{1,1}} \theta \cdots \frac{L_{1,1}L_{m,2} - L_{1,2}L_{m,1}}{L_{1,1}} \theta \cdots \frac{L_{1,1}L_{m,1} - L_{1,1}L_{1,m}}{L_{1,1}} \theta
\end{bmatrix}^{-1}
\]

where \( c_{2j} = \frac{L_{1,1}L_{j,j}}{L_{1,1}L_{j,j} - L_{1,j}L_{j,1}}. \) Continuing this process, we obtain

\[
R = \begin{bmatrix}
\frac{\tilde{L}_{1,1}N_1^{-1} \theta}{L_{1,1}} & \frac{\tilde{L}_{1,2}N_1^{-1} \theta \cdots \frac{\tilde{L}_{1,m}N_1^{-1} \theta m}{L_{1,1}}}{L_{1,1}} \\
\frac{\tilde{L}_{2,1} \theta \cdots \frac{\tilde{L}_{2,m} \theta}{L_{2,1}}}{L_{2,1}} & \frac{\tilde{L}_{2,2} \theta \cdots \frac{\tilde{L}_{2,m} \theta}{L_{2,1}}}{L_{2,1}} \\
\vdots & \vdots & \vdots \\
\frac{\tilde{L}_{m,1} \theta}{L_{m,1}} & \frac{\tilde{L}_{m,2} \theta \cdots \frac{\tilde{L}_{m,m} \theta}{L_{m,1}}}{L_{m,1}}
\end{bmatrix}
\]

Now we define

\[
P = \begin{bmatrix}
P_1 & 0 & \cdots & 0m \\
0 & P_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P_1
\end{bmatrix}
\]

Then

\[
PRP^{-1} = L^{-1}E \otimes \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.
\]
where $\otimes$ denotes the Kronecker product. Letting $\Omega$ be an $m \times m$ invertible matrix yields

$$(\Omega \otimes I) P R P^{-1}(\Omega \otimes I)^{-1} = (\Omega \otimes I)L^{-1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} (\Omega \otimes I)^{-1} = (KL^{-1}L)^{-1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{10}$$

Lemma 3- Let rank $A(x) = \text{const.}, \forall x \in I$ and the elements of $A(x)$ are in $C^p([0,1], \mathbb{R}^{r \times r})$. Then the initial value problem has only trivial solution, where $V(x)$ is defined by (5).

$$(V(x)y(x))^{(p)} + y(x) = 0, \quad y(0) = y'(0) = \cdots = y^{(p)}(0) = 0, \forall x \in I$$

3. Index notions

Dealing with DAEs or IAEs, an important concept is the use of index for existence and uniqueness of their solutions, and for their numerical analysis. The techniques for these investigations have important roles on the definitions of index, so there are different concepts of index. The connection between different index nominations has their importance in the study of DAEs or IAEs. We compare some effective definitions in this area which were given at the end of 20th century and we give some definitions that may be defined and their relations with each other.

Considering a DAE of the form

$$A(x)y'(x) + B(x)y(x) = q(x), \tag{11}$$

and we investigate it as an IAE of the form (1) by using $y(t) = x'(t)$

$$A(t)y(t) + \int_0^t B(t)y(s)ds = q(t) + x(0)B(t), \tag{12}$$

Definition 3- We say the differentiation index of the system (11) is $\nu$ (indd = $\nu$), if $\nu$ is the minimum possible number of differentiating (11) to obtain a system of the Ordinary Differential equations. Gear introduced the differentiation index using index reduction procedure to obtain existence and uniqueness conditions of this equations.

Definition 4- We say the differentiation index of the system (1) is $\nu$ (indd = $\nu$), if $\nu$ is the minimum possible number of differentiating (1) to obtain a system of the second kind Volterra integral equations. It is easy to check that (1) has differentiation index $\nu$ iff (11) has differentiation index $\nu$. This definitions of index are extended in two different way.

Definition 5- Suppose $\Phi_0 y = A(t)y + \int_0^t k(t,s)y(s)ds$ be the associated operator to the system

(1). If the $r \times r$ matrices $V_j(t)$ exist such that the operator

$$\Phi_0 y = \sum_{j=0}^{\nu} (V_j(t)) \frac{d^j(y)}{dt^j}, \quad t \in I,$$

satisfies in

$$(\Phi_0 o \Phi_0) y = y + \int_0^t \Phi_0 k(t,s)y(s) ds,$$
then $∅_v$ is called a Left Regularizing Operator (LRO). The minimum possible value of $v$ is called Left index of $∅_v$ for the system (1) and we write $\text{ind}_l = v$.

Definition 6- Suppose $∅_0y = A(t)y + \int_0^t k(t,s)y(s)ds$ be the associated operator to the system (1). If the $r \times r$ matrices $V_j(t)$ exist such that the operator $∅_v = d_j(V_j(t)y) + t$, $t \in I$, satisfies in

$$(∅_v∅_0)y = B(t)y + \int_0^t ∅_v k(t,s)y(s)ds,$$

for a nonsingular $r \times r$ matrix $B(t)$ on $I$, then $∅_v$ is called a Regularizing Operator (RO). The minimum possible value of $v$ is called operator index of $∅_v$ for the system (1) and we write $\text{indo} = v$. We show existence and uniqueness of the solution for system (1) by the following theorem.

Definition 7- Suppose $A \in \mathcal{C}^v(I, R^{r \times r})$, and $k \in \mathcal{C}^v(D, R^{r \times r})$. Let

$$A_0 \equiv A, \quad k_0 = k,$$

$$∅_iy = \frac{d}{dt} \left((I - A_i(t))A_i^{-}(t))y\right) + y$$

$$A_{i+1} = A_i + (I - A_i(t)A_i^{-}(t))k_i(t,t), \quad k_{i+1} = ∅_i k_i,$$

Then we say that the 'rank degree' index of $A_i$ is $v$ if $\text{rank} A_i(t) = \text{const}, \forall t \in I$, for $i = 0,1,...,v-1$, $\det A_i \neq 0$. Moreover, we say that the 'rank degree' index of system (1) is $v$ (indr = $v$) if in addition to the above hypotheses, we have $f \in \mathcal{C}^v(I, R^r)$. And $F_{i+1} = ∅_i F_i, \quad F_0 = f$.

Definition 8- The DAE in (11) is said to be of global index $v$ DAEs(indg = $v$ ), if there exist regular matrix functions $E \in \mathcal{C}(I, L(R^m))$ and $F \in \mathcal{C}^1(I, L(R^m))$ so that multiplying (11) by $E(t)$ and transforming $F(t)^{-1}x(t) = \tilde{x}$ lead to the decoupled system

$$[\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \tilde{x}'(t) + [W(t) 0] \tilde{x}(t) = E(t) q(t).]$$

(15)

where $J$ is a constant nilpotent Jordan block matrix, $\text{ind}(J) = v$. The projection type definition for DAEs which introduced by Marz and her collaboration can be extended to the IAEs, which can be found somehow at the works of Brunner and Pishbin. For given $A, B \in L(I, L(R^m))$, define the chain of matrix functions

$$A_0 = A, \quad B_0 = B - AP,$$

$$A_{i+1} = A_i + B_i Q_i, \quad (16)$$

$$B_{i+1} = (B_i - A_{i+1}(P_0 P_1 ... P_{i+1}')) P_i, \quad i \geq 0$$
Where $P_j = I - Q_j$ and $Q_j(t)$ projects onto $N_j(t) = \ker A_j(t)$, $t \in I, j \geq 0$.

Definition 9- The pair $\{A, B\}$ of continuous matrix functions (and also the DAE (11)) is said to be index- $\nu$ tractable (indt = $\nu$ ) if all matrices $A_j(t), t \in I, j = 0, \ldots, \nu - 1$ within the chain (16) are singular with smooth null space, and $A \nu(t)$ remains nonsingular on $I$.

Definition 10- The equation $F(Y', Y) = 0$ has perturbation index $\nu$ (indp = $\nu$ ) along a solution $Y$ on $I$ if $\nu$ is the smallest integer such that, for all functions $Y$ having a defect $F(Y', Y) = \delta(x)$, there exists on $I$ an estimate

$$\|F(x) - Y(x)\| \leq C \left( \|F(0) - Y(0)\| + \max_{x \in I}\|\delta(x)\| + \cdots + \max_{x \in I}\|\delta^{(\nu-1)}(x)\| \right)$$

whenever the expression on the right-hand side is sufficiently small. Here $C$ denotes a constant which depends only on $F$ and the length of the interval. The perturbation index defined by Hairer, Lubich and Roche [9] has important role in analyzing of numerical treatment of DAEs. The inequality should be replaced in above definition for IAEs.

$$\|F(x) - Y(x)\| \leq C \left( \max_{x \in I}\|\delta(x)\| + \cdots + \max_{x \in I}\|\delta^{(\nu)}(x)\| \right)$$

4. Collocation method for IAEs

Let

$$I_h = \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\},$$

be a given (not necessarily uniform) partition of $I$, and set $\sigma_n := [t_n, t_{n+1}], \sigma_n := [t_0, t_n + 1], \text{ with } h_n = t_{n+1} - t_n$ ($n = 0, 1, \ldots, N - 1$) and diameter $h = \max \{h_n : 0 \leq n \leq N\}$. Each component of the solution of (1) is approximated by elements of the piecewise polynomial space

$$\mathcal{I}_m^{(0)}(I_h) := \{v \in C(I) : v|_{\sigma_n} \in \pi_m(n = 0, 1, 2, \ldots, N - 1)\}, \quad (18)$$

where $\pi_m$ denotes the space of all (real valued) polynomials of degree not exceeding $m$. The collocation solutions $u_h \in \left(\mathcal{I}_m^{(0)}(I_h)\right)^r$ for (1) are defined by the equation

$$A(t)u_h(t) + \int_0^t k(t, s)u_h(s)ds = g(t), \quad (19)$$

for $t \in X_h = \{t_n \in \sigma_n : 0 = c_0 < c_1 < \cdots < c_m \leq 1, n = 0, 1, 2, \ldots, N - 1\}$

and the continuity conditions

$$u_{n-1}(t_n) = u_n(t_n), \quad n = 1, 2, \ldots, N - 1, \quad (20)$$

The collocation parameters $c_i$ completely determine the set of collocation points $X_h$. By defining $u_n = u_h|_{\sigma_n} \in (\pi_m)^r$, we have
\[ u_n(t_n + sh_n) = \sum_{j=0}^{m} L_j(s) U_{n,j}, \quad s \in (0,1), \quad (21) \]

\[ L_j(v) := \prod_{k=0, k \neq j}^{m} \frac{v - c_k}{c_j - c_k}, \quad j = 0, \ldots, m, \]

where the polynomials denote the Lagrange fundamental polynomials with respect to the distinct collocation parameters \( c_i \). By partitioning the domain of integral in (19) and changing of variables, we have

\[ A(t_{n,i})U_{n,i} + F_{n,i} + h \int_{0}^{c_i} k(t_{n,i}, t + sh_i)u_n(t_n + sh_n)ds = g(t_{n,i}), \quad (22) \]

where the lag terms are defined by

\[ F_{n,i} = h \sum_{l=0}^{n-1} \int_{0}^{1} k(t_{n,i}, t_n + sh_n)u_{l}(t_l + sh_l)ds. \]

By substituting from (21) in (22), for \( i = 1, \ldots, m \) and using the continuity conditions (20), we obtain the \( rm \times rm \) system

\[ A(t_{n,i})U_{n,i} + h \sum_{l=0}^{m} \int_{0}^{1} k(t_{n,i}, t_n + sh_n)L_j(s)U_{n,j}ds = -h \int_{0}^{c_i} k(t_{n,i}, t_n + sh_n)L_0(s)U_{n-1}(t_n)ds - F_{n,i} + g(t_{n,i}), \quad (23) \]

with

\[ F_{n,i} = h \sum_{l=0}^{n-1} \left( \sum_{j=1}^{m} \int_{0}^{1} k(t_{n,i}, t_l + sh_l)L_j(s)U_{l,j}ds + \int_{0}^{1} k(t_{n,i}, t_l + sh_l)L_0(s)U_{l-1}(t_l)ds \right). \quad (24) \]

By solving the system (23), approximate solution of (1) is determined at the collocation points and at \( t_{n+1} \) by
5. An existence and uniqueness theorem for the approximate solution

Theorem 1- Let the system (1) satisfy in the following conditions:

1. $A(t) \in C^1(I,R^{r \times r})$, $f(t) \in C^1(I,R^r)$, $k(t,s), k_i(t,s) \in C(D,R^{r \times r})$,
2. $\text{ind} = 1$

Then the approximate solution of fully discretized continuous collocation method for sufficiently small $h$, with distinct collocation parameters $c_0 = 0, c_1, \ldots, c_m \in (0,1]$ exists and is unique. This theorem is also true for continuous collocation method.

6. Difference inequalities

Lemma 6- (Gronwall inequality) Assume that $\{k_j\}$, $(j \geq 0)$ is a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \leq \rho_0$ and

$$\varepsilon_n \leq \rho_0 + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j \varepsilon_j$$

with $\rho_0 \geq 0$, $q_j \geq 0$, $(j \geq 0)$. Then

$$\varepsilon_n \leq (\rho_0 + \Sigma_{j=0}^{n-1} q_j) \exp(\Sigma_{j=0}^{n-1} k_j).$$

Lemma 7- Let $B_j$, $(j \geq 0)$ be a uniformly bounded sequence of $v \times v$ matrices, $M = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_v)$

Table 1 Computed errors $d_n$ for example 1.

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<tr>
<th>$d_n$</th>
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<tr>
<td>0/00471998</td>
<td>i=2</td>
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</tr>
<tr>
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<tr>
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<td>i=5</td>
</tr>
<tr>
<td>1/43508 \times 10^{-8}</td>
<td>i=6</td>
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</table>

Corresponding Author: Jalil Manafian
Email: j-manafian@iau-ahar.ac.ir
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### Table 2 Computed errors $d_n$ for example1.

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<tbody>
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<tr>
<td>0.000192775</td>
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<tr>
<td>$8.19504 \times 10^{-6}$</td>
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<td>$3.70787 \times 10^{-7}$</td>
<td>$i=5$</td>
</tr>
<tr>
<td>$1.33061 \times 10^{-8}$</td>
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### Table 3 Computed errors $d_n$ for example1.

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### Table 4 Computed errors $d_n$ for example1.

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<tbody>
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<td>0.0311737</td>
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<tr>
<td>0.000174</td>
<td>$i=6$</td>
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7. Conclusion

The main idea of this work was to give a simple method for solving the functional integral equations of Fredholm type and Volterra type. We carefully applied a reliable modification of based on Lagrange interpolation method for functional integral equations. The main advantage of this method is the fact that it gives the accurate approximate solution. We derive an expansion method to treat these equations. Finally, For showing efficiency of the method we give some numerical examples. It is shown that this method is a promising tool for some of the linear and nonlinear integral equations.

8. References